Exam
SOLID MECHANICS (NASM)
January 30, 2018, 09:00–12:00 h

This exam comprises four problems, for which one can obtain the following points:

Question	# points
1	1.5+1+0.5=3
2	1+2+2=5
3	1+2+0.5+1=4.5
~~4~~	~~1+1=2~~

The grade is calculated as $9 * (\# \text{ points})/$~~14.5~~ 12.5 $+ 1$.

Question 1 In 1913, Von Mises [1] proposed the following scalar representation of a stress tensor:

$$\sigma_{\mathrm{v}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'}\,,$$

where, as usual,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \left(\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}\right)\boldsymbol{I}$$

is the stress deviator.

a. Compute σ_{v} for uniaxial tension at a stress σ, as well as for hydrostatic compression with magnitude p.

b. Express σ_{v} in terms of the shear stress τ in the case of pure shear defined by: $\sigma_{12} = \sigma_{21} = \tau$, otherwise $\sigma_{ij} = 0$.

c. Finally, compute the Von Mises stress for combined state of uniaxial tension plus shear.

Question 2 The lecture notes have the Navier equation in three dimensions, i.e. Eq. (3.24), but there are also two-dimensional versions for planar problems.

a. In order to derive the plane-stress Navier equations, show that inversion of the plane-stress version of Hooke's law Eq. (3.41) leads to, cf. Eq. (5.3),

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}\left[\varepsilon_{\alpha\beta} + \frac{\nu}{1-\nu}\varepsilon_{\kappa\kappa}\delta_{\alpha\beta}\right],$$

where Greek indices run from 1 to 2.

b. Show that the Navier equations for states of plane stress read

$$u_{\alpha,\beta\beta} + \frac{1+\nu}{1-\nu}u_{\beta,\beta\alpha} = 0\,.$$

c. Prove that the plane-strain Navier equations can be obtained from the plane-stress version in (b) by replacing ν with $\nu/(1-\nu)$.

[1] Richard Edler von Mises (1883–1953) was an Austrian–Hungarian scientist who worked on fluid mechanics, aerodynamics, aeronautics, statistics and probability theory. Born in what now is Lviv (Ukrania), he lived and worked in Germany, Turkey and the US (Harvard University). He is best known in solid mechanics for his stress measure, which is the basis of an energy-based criterion for plasticity, formulated well in advance of crystal plasticity or even the notion of a dislocation.

Write name and student number on each page!

Question 3 A cantilever of length $L := a + b$ is loaded by a forces of magnitude F at the end of the beam and a bending moment of magnitude FL at a distance b from the clamp. The beam is made of a linear elastic material with modulus E, and has a moment of inertia I.

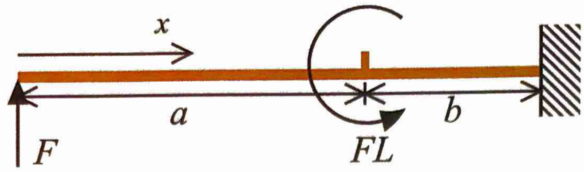


Figure 1: Cantilever loaded by force F and bending moment FL.

a. Determine the distribution of the bending moment $M(x)$ along the beam. Clearly indicate your sign convention.

b. The end force F tends to make the beam bend upwards, whereas the bending does the opposite. Use the "forget-me-nots"" from Fig. 3.6 to determine the value of b/L for which the deflection $w = 0$ at $x = 0$.

c. Irrespective of where the external moment FL is being applied, the bending moments $M(0) = 0$ and $M(L)) = 0$. This means that the external loading is equivalent to the one shown below. Show that this is correct indeed.

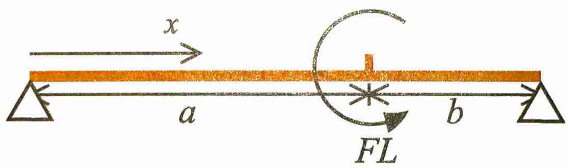
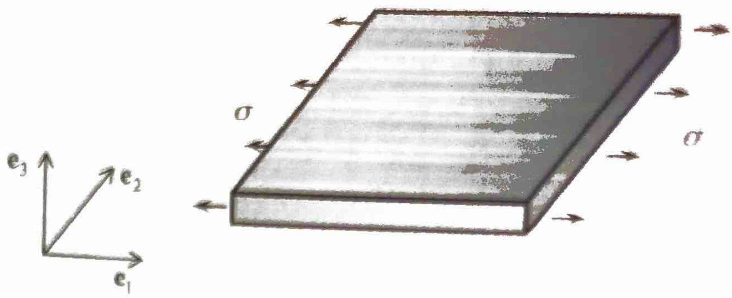


Figure 2: Simply supported beam loaded by force F and bending moment FL.

d. Note that in both Figs. 1 and 2, $w(0) = w(L) = 0$. Then, if the loading is the same for any value of b, why is it that the deflection $w(0)$ in Fig. 1 vanishes only for the particular value of b determined in (b)?

Write name and student number on each page!

Question 4 Consider a homogeneous plate subject to tension in the $\boldsymbol{e}_1$ direction, see figure. The plate is made of a single crystal but its orientation is unknown.



a. First assume a uniaxial stress state (i.e. plane stress in both $\boldsymbol{e}_2$- and $\boldsymbol{e}_3$- direction). Determine the most likely slip plane and slip direction, i.e. where the resolved shear stress is maximum.

b. Repeat the analysis in case the plate is very wide in the $\boldsymbol{e}_2$-direction so that it is in a state of plane strain tension (with $\varepsilon_{22} = 0$).

NB: the plate is a three-dimensional object.

Answers Exam
SOLID MECHANICS (NASM)
January 30, 2018, 09:00–12:00 h

Question 1

a. For uniaxial tension, $\sigma_\text{v} = |\sigma|$. [1]

For hydrostatic pressure: $\sigma_\text{v} = 0$ [1/2]

b. For pure shear, $\sigma_\text{v} = \sqrt{3}|\tau|$. [1]

c. For tension and shear: $\sigma_\text{v} = \sqrt{\sigma^2 + 3\tau^2}$ [1/2]

Question 2

a. Key step in inverting Eq. (3.41) is contraction over α and β to find

$$\sigma_{\kappa\kappa} = \frac{E}{1-\nu}\varepsilon_{\kappa\kappa}$$

Substituting this back into (3.41) and re-ordering gives the request result. [1]

b. The procedure is to substitute Hooke's law into the equilibrium conditions, Eq. (3.42), followed by insertion of the strain–displacement relations $\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$. [2]

c. Two elements: (i) replace ν with $\nu/(1-\nu)$ in the expression under (b), to obtain

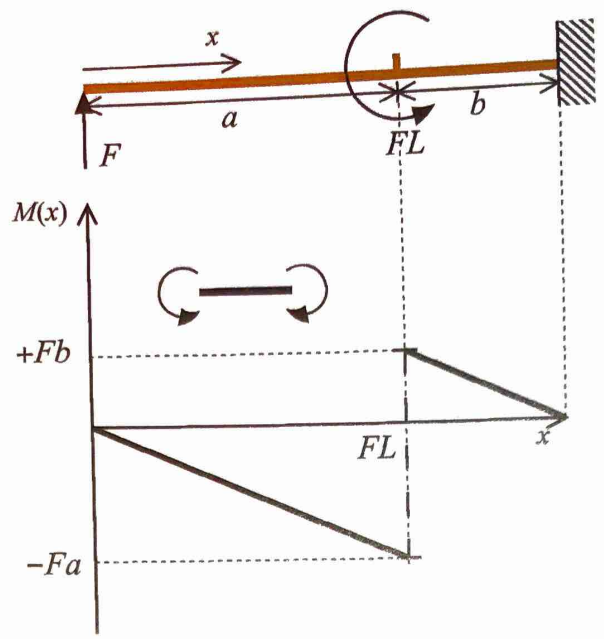
$$0 = u_{\alpha,\beta\beta} + \frac{1}{1-2\nu}u_{\beta,\beta\alpha}; \qquad (2)$$

followed by (ii) derivation of the same result starting directly from the plan-strain version of Hooke's law, Eq. (3.39). [2]

Question 3

a. [1]

Write name and student number on each page!



b. Temporarily writing M instead of FL for clarity,

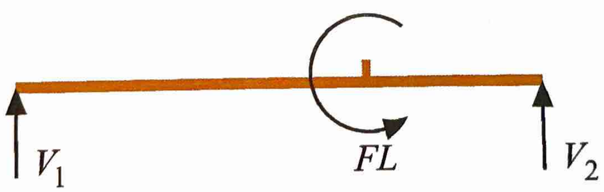
$$w(0) = \left(\frac{Mb^2}{2EI} + a\frac{Mb}{EI}\right) - \frac{FL^3}{3EI}.$$

After substitution of $M = FL$, solve b/L from $w(0) = 0$ to find

$$\frac{b}{L} = 1 - \frac{1}{\sqrt{3}}$$

1

c.



Two equilibrium equations for the unknown support forces V_1 and V_2:

$$V_1 + V_2 = 0, \quad V_1 L - (FL) = 0 \text{ (or, equivalently, } (FL) + V_2 L = 0)$$

with solution $V_1 = -V_2 = F$. This means that the left-hand support in Fig. 2 provides the same upward force as the external force on the left end of the cantilever in Fig. 1. QED 0.5

d. In general, the configuration in Fig. 2 will lead to a rotation of the beam at $x = L$, clockwise or counter-clockwise depending on b/L. It is only for $b/L = 1 - 1/\sqrt{3}$ that the rotation $w'(L)$ happens to be $= 0$, which is what it is in the configuration in Fig. 1. 1

Question 4

a. In uniaxial tension, the maximum shear stress is found at 45° relative to the $\boldsymbol{e}_1$ axis in the $\boldsymbol{e}_1$–$\boldsymbol{e}_2$ plane, but also at 45° relative to the $\boldsymbol{e}_1$ axis in the $\boldsymbol{e}_1$–$\boldsymbol{e}_3$ plane. These two determine the most likely slip planes. 1

b. When the plate is constrained in the $\boldsymbol{e}_2$ direction,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \nu\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\sigma > \nu\sigma > 0$, the largest shear stress (i.e. difference between principal stresses) is found in the $\boldsymbol{e}_1$–$\boldsymbol{e}_3$ plane. 1